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TECHNICAL NOTE

Robustness of Viability Controllers under Small Perturbations[1]

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Abstract. In this note, a perturbed control problem with state constraints depending on a parameter u is considered. Assuming that, for a certain value of u, there exists a viability controller, we explicitly estimate the range of variations of u for which the same controller gives viable solutions.

Key Words. Differential inclusions, state constraints, regulation maps.

1. Introduction

We consider a constrained control problem,

$$x'(t) = f(x(t), v(t)), \tag{1a}$$

$$v(t) \in V(x(t)), \tag{1b}$$

$$x(t) \in K, \tag{1c}$$

$$x(t_0) = x_0 \in K, \tag{1d}$$

where K is a given closed subset of a finite-dimensional space X and the set-valued map $V(\cdot)$ denotes a priori feedback. Relying on the modelled problem, we may require that either all solutions of (1) starting in K stay

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in K or there exists at least one solution with the same property. In the first case, K is called invariant; in the second case, it is called viable. Conditions ensuring invariance or viability for (1) are well known both for upper and lower semicontinuous feedbacks; see Refs. 1–3. In many cases, the dynamics as well as the constraints are not known with certainty, due to some unknown perturbations. We introduce the perturbed system

$$x'(t) = f(x(t), v(t)) + \xi_1(t), \tag{2a}$$

$$v(t) \in V(x(t)) + \xi_2(t), \tag{2b}$$

$$x(t) \in K(u), \tag{2c}$$

$$x(t_0) = x_0 \in K(u), \tag{2d}$$

where we assume that the noise modelled by $\xi_l(\cdot)$ is bounded, i.e., $\xi_l \in P_l(x)$, and the set-valued map $P_l(\cdot)$ has bounded values, $l = 1, 2$. Moreover, the set of constraints is not fixed, but it depends on an uncertain parameter u. A nonstochastic approach to robustness in control problems with disturbances was studied, for example, in Refs. 4 and 5. We consider the regulation map, i.e., the viability controller for (2), defined for $x \in K(u)$ to be the following set-valued map:

$$R(x, u) := \{v \in V(x) \mid f(x, v + \xi_2) + \xi_1 \in T_{K(u)}(x), \text{ for all } \xi_l \in P_l(x), l = 1, 2\}.$$

Let us assume that, for a certain parameter u_0, this map has nonempty values, i.e.,

$$R(x, u_0) \neq \emptyset, \qquad \text{for every } x \in K(u_0).$$

Under adequate regularity assumptions on the dynamics of (2), the above condition implies the existence of a solution for a fixed set $K(u_0)$. Since in some cases there may be fluctuations in the parameter u around u_0, we are interested to estimate the range of those perturbations which do not destroy the viability of the set $K(u)$. Moreover, we are interested to know for which u around u_0 the regulation map $R(x, u)$ may be obtained from $R(x, u_0)$. To get such an estimate, we will assume that the map f is Lipschitz and $K(u)$ is given by smooth constraints,

$$K(u) = \{x \in \mathbb{R}^n \mid g_1(x, u) \leq 0, \ldots, g_v(x, u) \leq 0\}.$$

Moreover, we assume that a strong tangential condition is satisfied for $K(u_0)$. Our result essentially depends only on the Lipschitz constants involved and on the bounds on $f(\cdot)$, $V(\cdot)$, $g_i(\cdot)$, $P_l(\cdot)$. The explicit form of the estimation given here improves a similar theoretical result of Ref. 6.

2. Main Results

We consider the system (2) and we assume that the set $K(\cdot)$, depending on the parameter $u \in \mathbb{R}^m$, is given by v differentiable constraints

$$K(u) := \{x \in \mathbb{R}^n \mid g_1(x, u) \leq 0, \ldots, g_v(x, u) \leq 0\}.$$

For $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, let

$$I(x, u) = \{i = 1, \ldots, v \mid g_i(x, u) = 0\};$$

if $x \in K(u)$, then $I(x, u)$ denotes the set of active constraints at the point $x \in K(u)$. Taking into account that $K(\cdot)$ is defined by constraints, we extend the regulation map $R(x, u)$ to all $x \in \mathrm{dom}(V)$ in the following way:

$$R(x, u) := \{v \in V(x) \mid \langle D_x g_i(x, u), f(x, v + \xi_2) + \xi_1 \rangle \leq 0,$$
$$\text{for every } \xi_l \in P_l, l = 1, 2, i \in I(x, u)\}.$$

If $x \in K(u)$, then it is well known (see Ref. 7, p. 57) that

$$T_{K(u)}(x) = \{w \in \mathbb{R}^n \mid \langle D_x g_i(x, u), w \rangle \leq 0, \forall i \in I(x, u)\}.$$

Consequently, the above definition of the regulation map agrees with the one given in the preceding section.

The following is our main result.

Proposition 2.1. Let $\Omega \subset \mathbb{R}^n$, $\mathscr{V} \subset \mathbb{R}^p$ be nonempty sets, Ω open. Let $f: \Omega \times \mathscr{V} \mapsto \mathbb{R}^n$ be L_f-Lipschitz, and let $\|f(x, v)\| \leq B_f$ for every $x \in \Omega$, $v \in \mathscr{V}$. Let $P_l: \Omega \rightsquigarrow \mathbb{R}^n$, $l = 1, 2$ be such that $\|P_l(x)\| \leq B_{P_l}$ for every $x \in \Omega$ and $V: \Omega \rightsquigarrow \mathscr{V}$ be L_V-Lipschitz. Let $g_i: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, i = 1, \ldots, v$, be differentiable functions and let $u_0 \in \mathbb{R}^m$, $\Delta > 0$, and $\epsilon > 0$ be such that:

(a) $\displaystyle\bigcup_{\|u - u_0\| \leq \Delta} K(u) \subseteq \Omega;$

(b) the vectors $\{D_x g_i(x, u) \mid i \in I(x, u)\}$ are linearly independent for all $u \in \mathbb{R}^m$, $\|u - u_0\| < \Delta$, $x \in K(u)$;

(c) $\|Dg_i(x_1, u_1) - Dg_i(x_2, u_2)\| \leq L_{g'}(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x_1, x_2 \in \Omega$, $u_1, u_2 \in B(u_0, \Delta)$, $i = 1, \ldots, v$;

(d) $\|Dg_i(x, u)\| \leq B_{g'}$, for all $\|u - u_0\| < \Delta$, $x \in \Omega$, $i = 1, \ldots, v$;

(e) $$\left\|\frac{\partial g_i}{\partial x_j}(x, u)\right\| \geq \eta > 0,$$

for all $\|u - u_0\| < \Delta$, $x \in \Omega$, $i = 1, \ldots, v$, $j = 1, \ldots, n$;

(f) for every $x_0 \in K(u_0)$, there exists $v_0 \in V(x_0)$ such that, for every $i \in I(x_0, u_0)$,

$$\langle f(x_0, v_0), D_x g_i(x_0, u_0) \rangle \leq -\epsilon; \tag{3}$$

(g) $\epsilon > B_{g'}(L_f B_{P_2} + B_{P_1})$.

Choose $\delta > 0$ such that

$$\delta < \eta / (L_{g'}(B_{g'} + 1 + \eta)), \qquad \delta \leq \Delta,$$

and set

$$L = (B_{g'} + 1)/\eta + 1,$$
$$M = L(1 + 2L_{g'}\delta)/(1 - L_{g'}\delta L),$$
$$r = \delta(1 - L_{g'}\delta L)/(3(1 + L + L_{g'}\delta L)),$$

and

$$\delta_0 := \min\{r, [\epsilon - B_{g'}(L_f B_{P_2} + B_{P_1})]/[B_{g'} L_f M(1 + L_V) + B_f L_{g'}(M + 1)]\}. \tag{4}$$

Then, for every $u \in \mathbb{R}^m$ such that $\|u - u_0\| \leq \delta_0$, for every $x \in K(u)$, we have that:

(i) there exists $x_0 \in \Omega$ such that $\|x - x_0\| \leq M \|u - u_0\|$ and $I(x, u) \subseteq I(x_0, u_0)$;

(ii) $\{v \in V(x) \mid \|v_0 - v\| = d(v_0, V(x))$, where v_0 satisfies $(3)\} \subseteq R(x, u)$; consequently, if $V(\cdot) \equiv V$,

$$\{v \in V \mid \langle f(x_0, v), D_x g_i(x_0, u_0) \rangle \leq -\epsilon, i \in I(x_0, u_0)\} \subseteq R(x, u),$$

for every $\|u - u_0\| \leq \delta_0$, $x \in K(u)$; therefore, if $f(\cdot, V(\cdot) + P_2(\cdot)) + P_1(\cdot)$ is convex-valued and upper semicontinuous, (2) admits solutions for every $\|u - u_0\| \leq \delta_0$.

Remark 2.1. Note that, from (e), (g) and our choice of δ, it follows that $M < +\infty$, $r > 0$, $\delta_0 > 0$. Note also that (f) and (g) imply $R(x_0, u_0) \neq \emptyset$.

Remark 2.2. In Proposition 2.1, we may be interested in an optimal choice of δ which maximizes δ_0. For this, we set

$$r_1(\delta) = \delta(1 - L_{g'}\delta L)/(3(1 + L + L_{g'}\delta L)),$$
$$r_2(\delta) = [\epsilon - B_{g'}(L_f B_{P_2} + B_{P_1})]/[B_{g'} L_f M(1 + L_V) + B_f L_{g'}(M + 1)].$$

Let

$$A := 2LL_{g'}[B_{g'} L_f(1 + L_V) + B_f L_{g'}] - B_f L L_{g'}^2,$$
$$B := L[B_{g'} L_f(1 + L_V) + B_f L_{g'}] + B_f L_{g'} - 3L_{g'} L[\epsilon - B_{g'}(L_f B_{P_2} + B_{P_1})],$$
$$C := -3(1 + L)[\epsilon - B_{g'}(L_f B_{P_2} + B_{P_1})].$$

Solving the equation $r_1(\delta) = r_2(\delta)$, we get three solutions, two of which are positive,

$$\delta_1 = 1/(L_{g'}L),$$

$$\delta_2 = (-B + \sqrt{B^2 - 4AC})/(2A).$$

It is easy to see that from (g) it follows that r_2 is a strictly decreasing function, while r_1 has one local maxima for $\delta > 0$. Let δ_3 be the point in which r_1 is maximized, i.e.,

$$\delta_3 = (\sqrt{2 + 3L + L^2} - 1 - L)/(LL_{g'}).$$

Therefore, if $\delta_2 < \delta_3$, the optimal choice for δ is $\delta = \delta_2$, while if $\delta_2 \geq \delta_3$, then the optimal choice is $\delta = \delta_3$.

Proof of Proposition 2.1. Let $\bar{u} \in \mathbb{R}^m$ be such that $\|\bar{u} - u_0\| \leq \delta_0$, $\bar{y} \in \text{bd}(K(\bar{u}))$, and assume that $I(\bar{y}, \bar{u}) = \{1, \ldots, s\}$, $s \leq \nu$. To prove the assertion (i), we define the map $G: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s \times \mathbb{R}^m$,

$$G(y, u) := (g_1(y, u), \ldots, g_s(y, u), u).$$

Let

$$DG(y, u) = \begin{bmatrix} (D_x g_i(y, u))_{i=1}^s & (D_u g_i(y, u))_{i=1}^s \\ 0 & I \end{bmatrix} \tag{5}$$

be the Jacobian matrix of G. In order to show that the possibly multivalued inverse map G^{-1} is pseudo-Lipschitz around the point $(\bar{y}, \bar{u})$ (see Definition 1, p. 429, Ref. 8), we verify the assumptions of Theorem 4, p. 431, Ref. 8. Let $w \in \mathbb{R}^s \times \mathbb{R}^m$. By (b), there exists $v \in \mathbb{R}^n \times \mathbb{R}^m$ such that $w = DG(\bar{y}, \bar{u})v$. Moreover, we show that v can be chosen such that $\|v\| \leq L\|w\|$. Indeed, by (b) we may assume that the first s columns of the matrix $(D_x g_i(\bar{y}, \bar{u}))_{i=1}^s$ are linerly independent. Let

$$A(y, u) := \left(\frac{\partial g_i}{\partial x_j}(y, u)\right)_{i,j=1}^s.$$

Let us write $w = (w^s, w^m) \in \mathbb{R}^s \times \mathbb{R}^m$, $v = (v^s, v^{n-s}, v^m) \in \mathbb{R}^s \times \mathbb{R}^{n-s} \times \mathbb{R}^m$, i.e. $w^m = v^m$. Since

$$\text{rank}(A(\bar{y}, \bar{u})) = s \leq n,$$

we may assume that $v^{n-s} = 0$. Then,

$$\|v^s\| \leq \|A^{-1}(\bar{y}, \bar{u})\|(\|w^s\| + \|D_u g\| \, \|w^m\|).$$

Hence,

$$\|v\| \leq [\|A^{-1}(\bar{y}, \bar{u})\|(\|D_u g\| + 1) + 1]\|w\|.$$

From (e), it follows that $\|A(\bar{y}, \bar{u})\| > \eta$, i.e.,

$$\|A^{-1}(\bar{y}, \bar{u})\| \leq 1/\eta,$$

hence

$$\|v\| \leq L\|w\|. \tag{6}$$

Let now $(y, u) \in \Omega \times \mathbb{R}^m$ satisfy

$$\|y - \bar{y}\| + \|u - \bar{u}\| < \delta/2. \tag{7}$$

Let us note that (g) implies

$$\Delta/3 > \delta/3 > r \geq \delta_0 > 0, \qquad M > 0.$$

From (c), we get

$$\|DG(y, u) - DG(\bar{y}, \bar{u})\| \leq L_{g'}\delta. \tag{8}$$

For all $(y, u) \in \Omega \times \mathbb{R}^m$ satisfying (7) and for all $w \in \mathbb{R}^s \times \mathbb{R}^m$, there exists $v \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$w = DG(y, u)v + z,$$

where

$$z = DG(\bar{y}, \bar{u})v - DG(y, u)v.$$

From (6) and (8), it follows that

$$\|z\| \leq L_{g'}\delta\|v\| \leq L_{g'}\delta L\|w\|,$$

and by our choice of δ, $L_{g'}L\delta < 1$.

From Theorem 4, p. 431, Ref. 8, it follows that G^{-1} is pseudo-Lipschitz around $(\bar{y}, \bar{u})$; i.e., in particular for every $\|u - \bar{u}\| < r$, we have

$$G^{-1}(0, u) \cap ((\bar{y}, \bar{u}) + MrB) \neq \emptyset, \tag{9}$$

$$d(G^{-1}(0, u) \cap ((\bar{y}, \bar{u}) + MrB), G^{-1}(0, \bar{u}) \cap ((\bar{y}, \bar{u}) + 3MrB)) \leq M\|u - \bar{u}\|. \tag{10}$$

Let us define the set-valued map

$$u \mapsto Y(u) := \{y \in \mathbb{R}^n \mid g_1(y, u) = 0, \ldots, g_s(y, u) = 0\}.$$

Since

$$Y(u) = \pi_{R^n} \circ G^{-1}(0, u),$$

from (9), (10) it follows that

$$Y(u) \cap (\bar{y} + MrB) \neq \emptyset, \qquad \text{for } \|u - u_0\| < r,$$

and $Y(u)$ is pseudo-Lipschitz around $(\bar{y}, \bar{u})$.

Since $\delta_0 \le r$, we may choose $y_0 \in Y(u_0) \cap (\bar{y} + MrB)$ such that

$$\|\bar{y} - y_0\| \le d(Y(\bar{u}) \cap (\bar{y} + MrB),\ Y(u_0) \cap ((\bar{y} + 3MrB)) \\ \le M\|\bar{u} - u_0\|; \qquad (11)$$

since $y_0 \in Y(u_0)$,

$$I(y_0, u_0) \supseteq \{1, \ldots, s\};$$

thus, the claim (i) is proved.

Let $i \in I(\bar{y}, \bar{u})$. Choose y_0 as claimed above, and let $v_0 \in R(y_0, u_0)$ satisfy (3). Choose any point

$$\bar{v} \in \{V(\bar{y}) \mid \|v - v_0\| = d(v_0, V(\bar{y}))\}.$$

Then, by the fact that f is Lipschitz, (11), and (c)–(f) we have, for any $\xi_l \in P_l(\bar{y})$, $l = 1, 2$,

$$\begin{aligned}
&\langle f(\bar{y}, \bar{v} + \xi_2) + \xi_1, D_x g_i(\bar{y}, \bar{u})\rangle \\
&\le \langle f(\bar{y}, \bar{v} + \xi_2) + \xi_1 - f(y_0, v_0), D_x g_i(\bar{y}, \bar{u})\rangle \\
&\quad + \langle f(y_0, v_0), D_x g_i(y_0, u_0)\rangle + \langle f(y_0, v_0), D_x g_i(\bar{y}, \bar{u}) - D_x g_i(y_0, u_0)\rangle \\
&\le [L_f(\|\bar{y} - y_0\| + \|v - v_0\| + B_{P_2}) + B_{P_1}]B_{g'} - \epsilon \\
&\quad + B_f L_{g'}(\|\bar{y} - y_0\| + \|u - u_0\|) \\
&\le [L_f(M(1 + L_V)\|u - u_0\| + B_{P_2}) + B_{P_1}]B_{g'} - \epsilon \\
&\quad + B_f L_{g'}(M + 1)\|u - u_0\| \\
&\le \delta_0[B_{g'} L_f M(1 + L_V) + B_g L_{g'}(M + 1)] + B_{g'}(L_f B_{P_2} + B_{P_1}) - \epsilon \le 0,
\end{aligned} \qquad (12)$$

by (4). We proved that $\bar{v} \in R(\bar{y}, \bar{u})$. If V is constant, then $v_0 \in R(\bar{y}, \bar{u})$. This concludes the proof of the claim (ii) and Proposition 2.1. □

We now restate Proposition 2.1 in the case $K(u_0)$ is invariant.

Corollary 2.1. Let the assumptions of Proposition 2.1 hold. Suppose that assumption (f) is replaced by

(f′) for every $x_0 \in K(u_0)$, $i \in I(x_0, u_0)$, for every $v_0 \in V(x_0)$,

it holds that $\langle f(x_0, v_0), D_x g_i(x_0, u_0)\rangle \le -\epsilon$.

Then, if $\|u - u_0\| < \delta_0$, $R(y, u) = V(y)$ for any $y \in K(u)$. Consequently, if $P_1(\cdot)$ and $P_2(\cdot)$ are Lipschitz, then $K(u)$ is invariant.

Proof. Fix $\|\bar{u} - u_0\| \le \delta_0$, and let $\bar{y} \in K(\bar{u})$. Let y_0 be given by (i) of Proposition 2.1. Let $i \in I(\bar{y}, \bar{u})$ and $\bar{v} \in V(\bar{y})$. Choose $v_0 \in V(y_0)$ such that

$\|\bar{v} - v_0\| \leq L_V \|\bar{y} - y_0\|$. Then, applying the same argument of (12), we have

$$\langle f(\bar{y}, \bar{v} + \xi_2) + \xi_1, D_x g_i(\bar{y}, \bar{u}) \rangle \leq 0,$$

by our choice of δ_0. Hence, $R(\bar{y}, \bar{v}) = V(\bar{y})$. □

Example 2.1. Here, we give a simple example which shows how to use Proposition 2.1. Let us consider the following control system:

$$\dot{x}_1 = x_2 + v + \xi,$$

$$\dot{x}_2 = x_1 + v.$$

We set $x = (x_1, x_2)$ and

$$V(x) \equiv [-3, 3],$$

$$P_1(x) \equiv [-1, 1],$$

$$P_2(x) \equiv 0,$$

$$K(u) = [-1, 1] \times [-1, 1] + (u, 0), \qquad u \in [-1, 1],$$

i.e.,

$$K(u) = \{x \mid g_1(x, u) \leq 0, \ldots, g_4(x, u) \leq 0\},$$

$$g_1(x, u) = x_1 - u - 1,$$

$$g_2(x, u) = u - x_1 - 1,$$

$$g_3(x, u) = x_2 - 1,$$

$$g_4(x_2, u) = -1 - x_2.$$

Let

$$\Omega = (-3, 3) \times (-2, 2), \qquad \mathscr{V} = [-3, 3], \qquad \Delta = 1.$$

Then, keeping the notations of Proposition 2.1, we set

$$B_f = \sqrt{72}, \qquad L_f = 2, \qquad B_{g'} = \sqrt{2},$$

$$L_{g'} = 0, \qquad B_{P_1} = 1, \qquad B_{P_2} = 0,$$

$$L_V = 0, \qquad \eta = 1, \qquad \delta = 1.$$

By choosing $v = 3$ or $v = -3$, it is easy to see that, for every $x \in K(0)$, there exists $v \in V(x)$ such that, for $i = 1, \ldots, 4$,

$$\langle f(x, v), D_x g_i(x, 0) \rangle \leq -2 := -\epsilon.$$

Then,

$$L = M = \sqrt{2} + 2, \qquad r = 1/(3(3 + \sqrt{2})),$$

which gives

$$\delta_0 = r = 1/(3(3 + \sqrt{2})).$$

Thus, the same choice of $v = \pm 3$ provides solutions remaining in $K(u)$, for all $|u| < \delta_0$.

References

1. AUBIN, J. P., *Viability Theory*, Birkhäuser, Boston, Massachusetts, 1991.
2. AUBIN, J. P., and CELLINA, A., *Differential Inclusions*, Springer, Berlin, Germany, 1984.
3. BRESSAN, A., *Solutions of Lower Semicontinuous Differential Inclusions on Closed Sets*, Rendiconti del Seminario Matematico dell'Università di Padova, Vol. 69, pp. 99–107, 1983.
4. KURZHANSKI, A., *Control and Estimation under Uncertainty*, Nauka, Moscow, Russia, 1978 (in Russian).
5. LEITMANN, G., *Deterministic Control of Uncertain Systems via a Constructive Use of Lyapunov Stability Theory*, SIAM Conference on Control in the 90's, San Francisco, California, 1989.
6. KŘIVAN, V., *Perturbation of Viability Problem*, Journal of Mathematical Analysis and Applications, Vol. 155, pp. 131–139, 1991.
7. CLARKE, F., *Optimization and Nonsmooth Analysis*, Wiley, New York, New York, 1983.
8, AUBIN, J. P., and EKELAND, I., *Applied Nonlinear Analysis*, Wiley, New York, New York, 1984.