Set-valued Analysis and Differential Inclusions

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Projection of Differential Equations

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Abstract

In the paper a class of projections, called the G-projections, is defined. These projections are used to project the dynamics of a differential inclusion $\dot{x}(t) \in F(x(t))$ onto the contingent cone to a given set K. The existence of a solution to the projected differential inclusion is proven. The G-projections generalize the projection of the best approximation, and the G-projected differential inclusions were used to construct models in population biology.

1 Introduction

Projections of differential inclusions play an important role in many applications of mathematics. For example, projected differential inclusions were used in mechanics, see [11], or in economics to build planning models, see [5],[6],[7],[8]. In these applications the projection of the best approximation was used to project the dynamics of a differential inclusion or an equation onto the tangent cone to a given set. But in some cases the projection of the best approximation may not be adequate. For this reason, we define in this paper a class of projections, called the G-projections that generalize the projection of the best approximation. The G-projection of differential inclusions was motivated by the construction of population growth equations, see [9],[10]. Using the G-projection, the dynamics of a differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

is projected onto the contingent cone $T_K(x)$ to a given set K. We get a projected differential inclusion

$$\dot{x}(t) \in \Pi^G_{T_K}(F(x(t))) \tag{2}$$

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where $\Pi^G_{T_K}(F(x))$ denotes the G-projection of $F(x)$ onto the contingent cone $T_K(x)$. Unfortunately, the standard existence theorems for differential inclusions cannot be used to prove the existence theorem for (2), since in general, $\Pi^G_{T_K}(F(\cdot))$ does not inherit the properties of $F(\cdot)$ and K. To prove an existence theorem for (2), an existence theorem for a generalized differential variational inequality is given and it is proven that the projected differential inclusion (2) has the same solution set as this variational inequality. A similar approach to prove an existence theorem for projected differential inclusions in the case of the projection of best approximation was used in [1].

2 The G-projection

Definition 1. Let $A \subset \mathbf{R}^n$. Then $C_+(A)$ denotes the positive cone spanned by A, i. e.,

$$C_+(A) := \begin{cases} \bigcup_{k \geq 0} kA & \text{if } A \neq \emptyset \\ \{0\} & \text{if } A = \emptyset. \end{cases}$$

Remark. Let $g \in \mathbf{R}^n$. Instead of writing $C_+(\{g\})$ we will write $C_+(g)$.

Let us recall that for a non-empty set $A \subset \mathbf{R}^n$ the negative polar cone is $A^- := \{y \in \mathbf{R}^n \mid \langle y, a \rangle \leq 0, \text{ for every } a \in A\}$. For a set-valued map $F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ we denote by $\mathrm{Im}(F)$ its image, by $\mathrm{Dom}(F)$ its domain and by $\mathrm{Graph}(F)$ its graph.

Lemma 2.1 *Let $G : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ be a set-valued map with convex compact values.*

a) *Let $x_o \in \mathrm{Dom}(G)$ and $0 \notin G(x_o)$. Then $C_+(G(x_o))$ is the smallest closed convex cone containing the set $G(x_o)$. Consequently $C_+(G(x_o)) = (G(x_o))^{--}$.*

b) *Let $\mathrm{Graph}(G)$ be compact and $0 \notin \mathrm{Im}(G)$. Then the set-valued map $C_+(G) : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$ has a closed graph.*

Proof. a) Let $x_o \in \mathrm{Dom}(G)$. Since $C_+(G(x_o))$ is a cone spanned by a convex compact set disjoint from 0, it follows that it is closed. Therefore $C_+(G(x_o)) = (G(x_o))^{--}$, see [1,p.31].

b) We prove that $C_+(G(.))$ has closed graph by contradiction. Let $(x_n, y_n) \in \mathrm{Graph}(C_+(G))$, $(x_n, y_n) \longrightarrow (x, y) \notin \mathrm{Graph}(C_+(G))$. Since $y \notin C_+(G(x))$, i. e. $y \neq 0$, it follows that $y_n = 0$ only for a finite number of n's. Therefore we may assume that $x_n \in \mathrm{Dom}(G)$ for every n. It follows there exist $k_n \geq 0$ and $g_n \in G(x_n)$ such that $y_n = k_n g_n$. Since $G(.)$ has compact graph, we can choose a convergent subsequence from g_n (denoted again g_n) such that $g_n \longrightarrow g \in G(x)$. Since $0 \notin \mathrm{Im}(G)$ it follows that a subsequence of k_n is converging, $k_n \longrightarrow k < \infty$. Consequently, $y = kg$. Hence $y \in C_+(G(x))$, a contradiction.

Q.E.D.

In the following definition a class of projections, called the G-projections is defined. These projections "project" a point onto a set along the directions given by a set G.

Definition 2. Let $K \subset \mathbf{R}^n$ be a non-empty set, $G \subset \mathbf{R}^n$ be possibly empty. Then

1. For every $g \in G$ and every $u \in C_+(g) + K$ define

$$\begin{aligned} k_g^K(u) &:= \inf\{k \geq 0 \mid u - kg \in K\}, \\ \Pi_K^g(u) &:= u - k_g^K(u) g. \end{aligned}$$

2. Let $u \in C_+(G) + K$. Then

$$\Pi_K^G(u) := \bigcup_{\{g \in G \mid u \in C_+(g)+K\}} \Pi_K^g(u).$$

3. If $G = \emptyset$, then we set

$$\Pi_K^G(u) := u.$$

We say that $\Pi_K^G(u)$ is the G-projection of u onto the set K.

In the rest of this paper the set K from Definition 2 will be the contingent cone, see [1].

3 Generalized Variational Differential Inequalities

Let $K \subset \mathbf{R}^n$ be a non-empty closed set. Let us consider a set-valued map $F : K \rightsquigarrow \mathbf{R}^n$ and let

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset. \tag{3}$$

For upper semicontinuous, convex and compact valued map $F(\cdot)$, (3) is well known viability condition ensuring the existence of a viable solution (in K) for the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{4}$$

see [1],[2].

If (3) is not satisfied then the dynamics of (4) has to be changed at least on the set of these points where (3) does not hold in order to get a viable solution in K. Let $G : K \rightsquigarrow \mathbf{R}^n$ be given set-valued map and let us consider the following viability problem

$$\left.\begin{array}{rcl} \dot{x}(t) & \in & F(x) - C_+(G(x(t))) \text{ for almost all } \; t \in [0,T] \\ x(t) & \in & K \qquad \text{for every } t \in [0,T] \\ x(0) & = & x_0 \in K. \end{array}\right\} \tag{5}$$

Let

$$\Omega(x) := F(x) \cap (T_K(x) + C_+(G(x))), \quad x \in K. \tag{6}$$

Next theorem gives an existence result for (5).

Theorem 3.1 *Let $K \subset \mathbf{R}^n$ be a non-empty compact set, $F : K \rightsquigarrow \mathbf{R}^n$ be an upper semicontinuous set-valued map with non-empty compact convex values. Let $G : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with compact graph, convex values and $0 \notin \mathrm{Im}(G)$. Let $\mathrm{Dom}(\Omega) = K$ and*

$$\sup_{x \in K} \inf_{f \in \Omega(x)} \inf_{z \in \Pi^{G(x)}_{T_K(x)}(f)} \|f - z\| < c < \infty. \tag{7}$$

Then for every $T > 0$ there exists a solution to (5).

Proof. Let

$$M(x) := F(x) - (\bar{B}(0,c) \cap C_+(G(x))),$$

where $\bar{B}(0,c)$ denotes the closed ball with radius c, centered at 0. Obviously, $M(\cdot)$ has convex and compact values. The set-valued map

$$x \rightsquigarrow \overline{B}(0,c) \cap C_+(G(x))$$

is upper semicontinuous due to Lemma 2.1. Since $F(\cdot)$ is upper semicontinuous and K is compact, it follows (see [1, p.42]) that $\mathrm{Im}(F)$ and consequently $\mathrm{Im}(M)$ are compact sets. It is easy to see that $M(\cdot)$ has closed graph. Indeed, let $m_n \in M(x_n)$, $x_n \longrightarrow x, m_n \longrightarrow m$, i.e., $m_n = f_n - z_n$

where $f_n \in F(x_n), z_n \in \tilde{B}(0,c) \cap C_+(G(x_n))$. From compactness and upper semicontinuity it follows that we may choose subsequences converging to $f \in F(x)$ and $z \in \overline{B}(0,c) \cap C_+(G(x))$. Consequently $m = f - z \in M(x)$. Therefore $M(\cdot)$ is upper semicontinuous, having closed graph and compact image. Since

$$\forall x \in K, \quad \exists f \in \Omega(x), \ \exists z \in \Pi_{T_K(x)}^{G(x)}(f) \ \text{such that} \ \|f - z\| < c$$

then we have

$$z \in T_K(x) \cap \{f - (\overline{B}(0,c) \cap C_+(G(x)))\} \subset T_K(x) \cap M(x),$$

so that the tangential condition

$$M(x) \cap T_K(x) \neq \emptyset \tag{8}$$

is satisfied for every $x \in K$. The existence of a viable solution to

$$\dot{x}(t) \in M(x(t))$$

follows from the viability existence theorem, see [1], p.182.

Q.E.D.

Remark. Differential variational inequality, see [1]

$$\left.\begin{array}{rcl} \dot{x}(t) & \in & F(x(t)) - N_K(x(t)), \\ x(t) & \in & K \end{array}\right\} \tag{9}$$

where $N_K(x)$ denotes the normal cone, can be thought as a special case of (5) for $C_+(G(x)) := N_K(x)$ for every $x \in K$.

4 Projected Differential Inclusions

Let $K \subset \mathbf{R}^n$ be a non-empty set and let $F : K \leadsto \mathbf{R}^n$ be a set-valued map with non-empty values. Let $G : K \leadsto \mathbf{R}^n$ be a set-valued map and $\mathrm{Dom}(\Omega) = K$, where $\Omega(\cdot)$ is defined by (6). The G-projection of a differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{10}$$

is defined in the following way.

Definition 3. The G-projection of the differential inclusion (10) is the following differential inclusion

$$\dot{x}(t) \in \Pi^G_{T_K}(F(x(t))) := \bigcup_{f \in \Omega(x(t))} \Pi^{G(x(t))}_{T_K(x(t))}(f). \tag{11}$$

This differential inclusion is called the *projected differential inclusion.*

The following theorem shows that solutions to the differential inclusion (11) are solutions to the differential inclusion (5) and conversely.

Let us recall that a set $K \subset \mathbf{R}^n$ is *regular* if Bouligand contingent cone and Clarke tangent cone coincide and, consequently, they are convex cones, see [4]. If $x \rightsquigarrow T_K(x)$ is lower semicontinuous then the set K is regular, see [3]. In [3] such sets were called *sleek* sets.

Theorem 4.1 *Let $K \subset \mathbf{R}^n$ be a non-empty regular set. Let $F : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with non-empty values, $G : K \rightsquigarrow \mathbf{R}^n$. Let* $\mathrm{Dom}(\Omega) = K$ *and for every $x \in K$,*

$$G(x) \cap T_K(x) = \emptyset. \tag{12}$$

Then the viable solutions to the differential inclusion (5) are the solutions to the differential inclusion (11) and conversely.

Proof. Since

$$\Pi^G_{T_K}(F(x(t))) \subseteq F(x(t)) - C_+(G(x(t)))$$

then solutions to the differential inclusion (11) are solutions to the differential inclusion (5).

Conversely: every viable solution to the differential inclusion (5) is a solution to the differential inclusion (11). Let $x(\cdot)$ be a viable solution to the differential inclusion (5) on $[0,T]$, $(T > 0)$ i.e.

$$\dot{x}(t) = f(t) - z(t) \quad \text{for a.a. } t \in [0,T],$$

where $f(t) \in F(x(t)), z(t) \in C_+(G(x(t)))$. Let $[0,T] = E_1 \cup E_2$ where $t \in E_1$ if $G(x(t)) = \emptyset$ and $t \in E_2$ if $G(x(t)) \neq \emptyset$. Let us assume that $x(.)$ is not a solution to the differential inclusion (11). It means that there exists a set $\Lambda \subseteq [0,T]$ of a positive Lebesque measure $\mu(\Lambda) > 0$ such that

$$\dot{x}(t) \notin \Pi^G_{T_K}(F(x(t))) \quad \text{for} \quad t \in \Lambda$$

Since for $t \in E_1$, $\dot{x}(t) = f(t) \in \Pi^G_{T_K}(F(x(t)))$ it follows

$$\begin{aligned} \mu(\Lambda \cap E_1) &= 0, \\ \mu(\Lambda \cap E_2) &= \mu(\Lambda) > 0. \end{aligned}$$

For almost all $t \in \Lambda \cap E_2$,

$$\dot{x}(t) = f(t) - k(t)g(t) \in T_K(x(t))$$

where $g(t) \in G(x(t))$ and $k(t) > k^{T_K(x(t))}_{g(t)}(f(t))$. For almost all $t \in \Lambda \cap E_2$

$$-\dot{x}(t) = k(t)g(t) - f(t) \in T_K(x(t)).$$

Since $f(t) - k^{T_K(x(t))}_{g(t)}(f(t))g(t) \in T_K(x(t))$ for $t \in \Lambda \cap E_2$ and $T_K(x(t))$ is convex (since K is regular), then

$$(k(t) - k^{T_K(x(t))}_{g(t)}(f(t)))g(t) \in T_K(x(t)) \text{ for a.e. } t \in \Lambda \cap E_2.$$

Due to the assumption (12), the inequality $k(t) - k^{T_K(x(t))}_{g(t)}(f(t)) > 0$ cannot hold, i. e., $k(t) = k^{T_K(x(t))}_{g(t)}(f(t))$ for almost all $t \in \Lambda \cap E_2$ and $x(\cdot)$ is a solution to the differential inclusion (11).

Q.E.D.

5 Projected Differential Inclusions on the Sets Defined by Constraints

In this section it is assumed that the set K is defined by p functions $r_i(\cdot)$, $i = 1, \ldots, p$,

(13) $$K := \{x \in \mathbf{R}^n \mid r_1(x) \leq 0, \ldots, r_p(x) \leq 0\}.$$

We consider again differential inclusion (10). For K defined by (13) we may define a set-valued selection from $\Pi^G_{T_K}(F(x))$, denoted by $\pi(F(x))$ such that the solutions to

$$\dot{x}(t) \in \pi(F(x(t)))$$

are still solutions to (5) and conversely.

Throughout this section it is assumed that $r_i(\cdot)$, $i = 1, \ldots, p$ are strictly differentiable, see [4]. This is for example satisfied if $r_i(\cdot)$, $i = 1, \ldots, p$

are continuously differentiable. By $r_i'(x)$ we denote the strict derivative of $r_i(x)$. Let

$$I(x) := \{i = 1, \ldots, p \mid r_i(x) = 0\}.$$

If we assume that $r_i'(x)$, $i \in I(x)$ are positively linearly independent then it follows from [4] that

$$T_K(x) = \{v \in \mathbf{R}^n \mid \langle r_i'(x), v\rangle \leq 0, \ \ i \in I(x)\}.$$

Let $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$ be given set-valued maps. For every $x \in K$ we define:

$$G(x) := \text{conv}\{G_i(x) \mid i \in I(x) \ \text{such that}\ x \in \text{Dom}(G_i)\}. \tag{14}$$

Let

$$\omega(x) := \{f \in F(x) \mid \exists z \in C_+(G(x)), \langle r_i'(x), f - z\rangle = 0, i \in I(x)\}. \tag{15}$$

Now we define a set-valued map $\pi(F(.))$.

Definition 4. Let $K \subset \mathbf{R}^n$ defined by (13) be a non-empty set where the functions $r_i(.)$, $i = 1, \ldots, p$ are strictly differentiable. Let for every $x \in K$, $r_i'(x)$, $i \in I(x)$ be positively linearly independent and $F : K \rightsquigarrow \mathbf{R}^n$, $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$ be set-valued maps. Then for all $x \in K$, we define:

i) Let $f \notin T_K(x)$ and $f \in \omega(x)$ then

$$\pi(f) := \{f - z \mid z \in C_+(G(x)), \langle r_i'(x), f - z\rangle = 0, i \in I(x)\}$$

ii) Let $f \notin T_K(x)$, $f \notin \omega(x)$ and $f \in \Omega(x)$. Let p be any element of $\Pi^G_{T_K}(f)$. Then

$$\pi(f) := p.$$

iii) Let $f \in T_K(x)$ then

$$\pi(f) := f.$$

iv)

$$\pi(F(x)) := \{\pi(f) \mid f \in \Omega(x)\}.$$

Theorem 4.1 may be reformulated for the differential inclusion

$$\dot{x}(t) \in \pi(F(x(t))). \tag{16}$$

Theorem 5.1 *Let $K \subset \mathbf{R}^n$ defined by (13) be a non-empty set. Let $F : K \rightsquigarrow \mathbf{R}^n$ be a set-valued map with non-empty values, $G_i : K \rightsquigarrow \mathbf{R}^n$, $i = 1, \ldots, p$. Let $G(x) \cap T_K(x) = \emptyset$ for every $x \in K$ and $Dom(\Omega) = K$. Then the solutions to the differential inclusion (16) are the viable solutions to the differential inclusion (5) and conversely. Moreover, if $F(x) = \{f(x)\}$ is single valued and for every $x \in K$ and every $f \in F(x) \setminus T_K(x)$,*

$$(17) \qquad (C_+(G(x)) - C_+(G(x))) \cap T_K(x) \cap (-T_K(x)) = \{0\}$$

then $\pi(f(x))$ is single-valued.

Proof. First we prove that for all $x \in K$

$$(18) \qquad \pi(F(x)) \subseteq \Pi^G_{T_K}(F(x)).$$

Let us assume that there exist $f \in \Omega(x)$, $k > 0$, $g \in G(x)$ such that $f - kg \in \pi(f)$ and $f - kg \notin \Pi^G_{T_K}(F(x))$. Consequently, $f \notin T_K(x)$ and $f \in \omega(x)$, otherwise (18) is obviously satisfied. It follows $k > k_g^{T_K}$. Since $\langle f - kg, r_i'(x)\rangle = 0$ for $i \in I(x)$ and $\langle f - k_g^{T_K} g, r_i'(x)\rangle \leq 0$ for every $i \in I(x)$ we get

$$\langle (k - k_g^{T_K})g, r_i'(x)\rangle \leq 0, \quad i \in I(x).$$

Therefore $(k - k_g^{T_K})g \in T_K(x)$. This contradicts with the assumption $G(x) \cap T_K(x) = \emptyset$. Therefore $k = k_g^{T_K}$ and $\pi(F(x)) \subseteq \Pi^G_{T_K}(F(x)) \subset F(x) - C_+(G(x))$. It follows that the solutions to the differential equation (16) are the solutions to the differential inclusion (5).

Following the lines of the proof of Theorem 4.1, we assume that for a solution $x(.)$ to (5) there exists a set Λ of a positive Lebesque measure such that

$$\dot{x}(t) \notin \pi(F(x(t))), \quad t \in \Lambda.$$

Let E_1 and E_2 be as in the proof of Theorem 4.1. Since for $t \in E_1$, $\dot{x}(t) = f(t)$ and for almost all $t \in E_1$, $f(t) \in T_K(x(t))$, it follows $f(t) \in \pi(f(t))$ for almost all $t \in E_1$. Consequently, $\mu(\Lambda \cap E_1) = 0$ and $\mu(\Lambda \cap E_2) > 0$. Since $x(t)$ is viable, for almost all $t \in \Lambda \cap E_2$

$$(19) \qquad \dot{x}(t) = f(t) - k(t)g(t) \in T_K(x(t))$$

where $f(t) \in F(x(t))$, $g(t) \in G(x(t))$ and $k(t) > 0$. For almost all $t \in \Lambda \cap E_2$

$$(20) \qquad -\dot{x}(t) = k(t)g(t) - f(t) \in T_K(x(t)).$$

From (19), (20) it follows that for almost all $t \in \Lambda \cap E_2$,

$$\langle r_i'(x(t)), f(t) - k(t)g(t)\rangle = 0, \quad \text{for every} \quad i \in I(x(t)).$$

It follows that for almost all $t \in \Lambda \cap E_2$, $f(t) \in \omega(x(t))$ and $f(t) - k(t)g(t) \in \pi(f(t))$. Consequently, $x(t)$ is solution to (16).

We prove that for the single valued map $F(x) = \{f(x)\}$, $\pi(f(x))$ is single-valued too. Since for $f \in T_K(x)$ the statement is trivial, let us assume that $f \notin T_K(x)$. We may assume that $f \in \omega(x)$; otherwise $\pi(f)$ is single-valued. Let

$$z^1, z^2 \in \pi(f), \quad z^1 \neq z^2.$$

From Definition 4 it follows

$$z^1 = f - k^1 g^1, \quad z^2 = f - k^2 g^2,$$

where

$$k^1, k^2 > 0, \quad g^1, g^2 \in G(x),$$

and

$$\langle r_i'(x), f - k^1 g^1 \rangle = \langle r_i'(x), f - k^2 g^2 \rangle = 0, \quad i \in I(x).$$

Therefore,

$$\langle r_i'(x), k^1 g^1 - k^2 g^2 \rangle = 0, \quad i \in I(x)$$

and consequently

$$k^1 g^1 - k^2 g^2 \in T_K(x) \cap -T_K(x).$$

From the assumptions it follows

$$k^1 g^1 = k^2 g^2,$$

i.e., $\pi(f)$ is single valued.

Q.E.D.

Let us note that (16) generalizes the following differential inclusion

(21) $$\dot{x}(t) \in \pi^{\perp}(F(x(t)))$$

where $\pi^{\perp}$ denotes the projection of best approximation onto $T_K(x)$.

Proposition 5.1 *Let $K \subset \mathrm{R}^n$ defined by (13) be a non-empty set where the functions $r_i(.)$, $i = 1, \ldots, p$ are strictly differentiable and $r_i'(x)$, $i \in I(x)$ are positively linearly independent for every $x \in K$. Let $G_i(x) := \{r_i'(x)\}$ if $r_i(x) = 0$. Then (16) has the same solution set as (21).*

Proof. Let us note that under the assumptions it follows that the normal cone to K is

$$N_K(x) = \{ \sum_{i \in I(x)} \alpha_i r_i'(x) \mid \alpha_i \geq 0\}$$

and $T_K(x)$ is convex, see [4], p.57. From Theorem 5.1 it follows that (16) has the same solution set as (5) which is

$$\dot{x}(t) \in F(x(t)) - N_K(x(t)).$$

From [1] it follows that this differential inclusion has the same solution set as (21).

Q.E.D.

6 G–projection of control systems

The method of G–projection can also be used to "correct" the dynamics of control systems when there is no control regulating a viable solution. Let us consider a control system

$$\begin{array}{rcl} x'(t) & = & f(x(t), u(t)) \\ u(t) & \in & U \\ x(t) & \in & K, \end{array} \tag{22}$$

where $U \subset \mathbf{R}^l$, $K \subset \mathbf{R}^n$, $f : K \times U \mapsto \mathbf{R}^n$.

From the Filippov Lemma, [1,p.91] it follows that for continuous function $f(\cdot,\cdot)$, the control system (22) is equivalent to the following constrained differential inclusion

$$\begin{array}{rcl} \dot{x}(t) & \in & F(x(t)) := \{f(x(t), u(t)) \mid u(t) \in U\} \\ x(t) & \in & K. \end{array}$$

Consequently, the results from the previous sections may be reformulated for the control system (22).

Let us remark that using the contingent derivative instead of the contingent cone (see [1]) all the results stated for the autonomous case here may be reformulated to the non-autonomous case.

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